IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS, VOL. 37, NO. 9, SEPTEMBER 1990

The magnitude response of the Davidon-Fletcher-Powell method. The filter orders are determined from the frequency response by (9) with $L = M = N = 21$. It is decomposed by the outer product expansion, and the first four singular values are retained and the others are truncated. The specifications of 1-D digital filters are approximated to four singular values are retained and the others are truncated. It is decomposed by the outer product expansion, and the first four singular values are retained and the others are truncated. 

VII. NUMERICAL EXAMPLE

A numerical example is shown for the frequency-domain design. The frequency response of a cone filter [2] is the following:

$$H_d(\omega_1, \omega_2, \omega_3) = \begin{cases} 1 & \text{if } \omega_1 \leq 0.8\omega_3 \\ 1 - \frac{\omega_1 - 0.8\omega_3}{0.4\pi} & \text{if } 0.8\omega_3 < \omega_1 < 0.8\omega_3 + 0.4\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_1 = (\omega_1^2 + \omega_2^2)^{1/2}. \quad (24)$$

The magnitude response $A_d(l, m, n)$ on sampled points is produced from the frequency response by (9) with $L = M = N = 21$. It is decomposed by the outer product expansion, and the first four singular values are retained and the others are truncated. The specifications of 1-D digital filters are approximated to minimize the $l_2$ norm of the error of the magnitude response by the Davidon-Fletcher-Powell method. The filter orders are second for $\phi_k(z_1)$ and $\chi_k(z_2)$ ($p = 1, 2, 3, 4$), and third for $\phi_k(z_3)$ ($p = 1, 2, 3, 4$), respectively.

The magnitude responses of the specification and the approximation result on the plane with $\omega_1 = 0$ are shown in Fig. 2(a) and (b). The approximation error is 13.16% for the following criterion:

$$E_2 = \left[ \frac{\sum_{l,m,n} (A_d(l, m, n) - A'(l, m, n))^2}{\sum_{l,m,n} (A_d(l, m, n))^2} \right]^{1/2}. \quad (25)$$

VII. CONCLUSION

This paper has proposed the efficient design method of 3-D digital filters by the outer product expansion. It can decompose design problems of 3-D digital filters into design problems of 1-D digital filters by the outer product expansion. Both space domain specifications and frequency domain specifications can be designed by this method. Diagonal symmetries of 3-D digital filters can be exploited to reduce computations in the design procedure. Moreover, the parallel separable structure produced by this method has high parallelism, regularity, and modularity, and so it is suitable for parallel and VLSI implementation of 3-D digital filters.

ACKNOWLEDGMENT

The authors would like to thank Prof. Tatsuo Higuchi and Prof. Takayasu Ito of Tohoku University for valuable advice.

REFERENCES


Extension of the Open-Circuit Time-Constant Method to Allow for Transcapacitances

R. M. FOX AND S. G. LEE

Abstract—This paper reviews a previously published method for determining frequency-domain transfer functions of linear circuits and extends the method to allow for transcapacitors. The method is an extension of the familiar open-circuit time constant analysis technique, which depends on successive analyses of frequency-independent circuits. Where the original technique required finding the resistance "seen" by each capacitor, the extended technique requires finding a transresis-

Manuscript received May 26, 1989; revised January 8, 1990. This work was supported by the Semiconductor Research Corporation under Contract 88-SP-087 and by the Florida High Technology and Industry Council. This paper was recommended by Associate Editor C. A. T. Salama.

The authors are with the Department of Electrical Engineering, University of Florida, Gainesville, FL 32611. IEEE Log Number 9037136.
tance for each transcapacitor. This paper presents a derivation of the
generalized time-constant technique and demonstrates its application in
a simple circuit.

I. INTRODUCTION

The earliest small-signal models for bipolar and field-effect
transistors included only resistors, capacitors, and frequency-

independent controlled sources. These models were simple and
intuitive, and a number of useful "tricks" were available to
analyze circuits based on them. However, several workers [1]–[3]
pointed out inaccuracies due to these representations; more
accurate models were then developed which stored transistor
charges as state variables.

A key feature of such charge-based models is that the termi-
nal currents include terms of the form \( C_j \frac{dV_j}{dt} \), where \( C_j \)
is a constant and \( V_j \) is the instantaneous voltage at port \( j \). Such a
term can be represented in an equivalent circuit as a voltage-
controlled current source across port \( j \). In the small-signal
frequency domain, such a source takes on the value \( sC_j V_j \). \( C_j \)
has units of capacitance, but for \( j \neq k \), the controlled source
represents not a capacitor but a transcapacitor. A capacitor can
only be used when \( j = k \). In general, transcapacitances are
nonreciprocal, meaning \( C_{jk} \neq C_{kj} \). The need for nonreciprocal
elements to model active devices is not surprising—a trans-
capacitor is related to a capacitor the same way a transconduc-
tance is to a conductance.

A number of small-signal models including transcapacitors
has been implemented in circuit simulators such as SPICE.
These include the BISI MOSFET model [4], the bipolar tran-
sistor model in the device/circuit simulator MMSPIE [5], and
the five-terminal silicon-on-insulator model of Fossum and
Veeraraghavan [6]. Typically, these models are accurate to fre-
quencies about three times higher than those using only recipro-
cal capacitors. One would thus expect these models to be
popular with circuit designers. In fact, some designers have
resisted using these models, in part because some of the familiar
circuit analysis techniques cannot be applied.

In particular, the open-circuit time-constant method [7] for
estimating dominant pole frequencies has not been applicable
to circuits with transcapacitors. The present work demonstrates
how this simple method can be extended to such circuits. Fur-
thermore, this paper shows how to handle transcapacitors in an
extension of the time-constant technique [8], which allows sim-
plified calculation of complete transfer functions. It uses suc-
cessive analyses of resistive networks to find dc driving-point
and transfer functions, requiring no complex algebra or frequency-
dependent terms.

The extended time-constant method is derived in Section II.
The derivation is an extension of the development in [7]. This
method could be extended further to allow for inductances and
for other frequency-dependent controlled sources, but such ex-
tensions are excluded here. Section III gives an example of use
of the method.

II. DERIVATION OF THE EXTENDED
TIME-CONSTANT METHOD

Consider a multiport network, with a port defined for each
capacitor, for each transcapacitor, and for each voltage which
controls a transcapacitor (a transcapacitive controlling voltage,
or TCV), as well as ports for the output (port \( o \)) and for the
input (port \( i \)). Now define \( \Delta \) as the determinant of the short-circuit
admittances \( Y_{jk} \). It is convenient to define \( N \) as one less
than the number of ports. \( \Delta \) has the form

\[
\Delta = \begin{vmatrix}
E_{11} + sC_{11} & E_{12} + sC_{12} & \cdots & E_{1,N+1} + sC_{1,N+1} \\
E_{21} + sC_{21} & E_{22} + sC_{22} & \cdots & E_{2,N+1} + sC_{2,N+1} \\
\vdots & \vdots & \ddots & \vdots \\
E_{N+1,1} + sC_{N+1,1} & E_{N+1,2} + sC_{N+1,2} & \cdots & E_{N+1,N+1} + sC_{N+1,N+1}
\end{vmatrix}
\]

(1)

where \( E_{kl} \) is the value of the capacitor (if any) at port \( j \), and \( C_{jk} \), \( j \neq k \),
is the transcapacitance from a TCV at port \( j \) that controls a
transcapacitor at port \( k \). No capacitive element occurs in more
than one \( Y_{jk} \). In most circuits most of the possible coefficients
\( C_{jk} \) are zero.

All possible transfer functions can be expressed as ratios of
cofactors formed by deleting certain rows and columns of \( \Delta \). It
is not necessary in practice to actually form these determinants;
they are used here only for the purposes of this derivation. The
numerator cofactor \( \Delta_N \) in each case is the \( N \)-th order
determinant formed by deleting row \( i \) for the input and column
\( o \) for the output. All terms \( Y_{jk} \) having \( j = i \) or \( k = o \) are thereby
deleted. Such deleted terms correspond to self-admittances at
the input and output ports, and reverse transadmittances from
the output back to some other port and from any port back to
the input. It is useful to define \( C_N \) as the set of all nonzero
capacitive elements \( C_{jk} \), having \( j \neq o \) and \( k \neq i \), corresponding
to the coefficients of \( s \) in \( \Delta_N \).

The denominator determinant \( \Delta_D \) is formed in a similar way.
Selection of rows and columns to be deleted depends on the
type of transfer function to be computed. Simply, any port
where the short-circuit output current is defined or where the
input voltage is applied is considered as a short circuit in finding
\( \Delta_D \) and its corresponding row and column are deleted.
The various possibilities are summarized in Table I. \( \Delta_D \)
is the cofactor of \( \Delta \) with row and column deleted for any shorted port.
The nonzero coefficients of \( s \) in \( \Delta_D \) form the set \( C_D \); the set of
all nonzero capacitive elements \( C_{jk} \) for which neither \( j \) nor \( k \)
corresponds to a shorted port.

When expanded, \( \Delta_D \) has the form

\[
\Delta_D = b_0 + b_1 s + b_2 s^2 + \cdots + b_N s^N
\]

(2)

where \( N_d \) is the order of \( \Delta_D \). \( N_d \) equals \( N \) for a voltage gain or
a current gain, \( N - 1 \) for a transadmittance, or \( N + 1 \) for a
transimpedance. Now, \( b_0 = \Delta_D^0 \), the determinant evaluated with
all capacitive elements set to zero. It is useful to write

\[
\Delta_D \Delta_D^0 = 1 + a_1 s + a_2 s^2 + \cdots + a_N s^N
\]

(3)

where \( a_j = b_j / \Delta_D^0 \).

Expanding the determinant shows that the first-order term is

\[
a_1 = \sum_{C_D} \left[ C_{jk} \frac{\Delta_D N_{jk}}{\Delta_D^0} \right]
\]

(4)
where the summation over \( C_D \) implies inclusion of a term for each element in \( C_D \). From Cramer’s Rule, \( a_{jk} / \Delta_D = R_{jk}^0 \), the transresistance from a current source at port \( j \) to an open-circuit voltage at port \( k \), with all capacitive elements set to zero. For a capacitor \( C_{jk} \), the corresponding element \( R_{jk}^0 \) is just the driving-point resistance at port \( j \). These resistive coefficients can be calculated from straightforward circuit analysis, so the determinants themselves are not needed. Since the sum of these RC products forms a conservative estimate of the circuit’s dominant time constant, this method is often called “time-constant analysis.”

Calculation of the second-order coefficient \( a_z \) is more complex. Second-order terms involve products of pairs of capacitive elements. Let \( C_{ij} \) denote the set of all unique products of pairs \( C_{jk}, C_{im} \) of nonzero capacitive elements in \( C_D \) such that \( j \neq i \) and \( k \neq m \). This set contains all the pairs except those where a capacitor and a transcapacitor, or two transcapacitors, exist at the same port (coefficients in the same row) or a capacitor and a TCV, or two TCVs (same column), exist at the same port. Each element in this set is multiplied by a corresponding coefficient

\[
a_{jk, lm} = \frac{\Delta^0_{D,jk,lm}}{\Delta_D^0}.
\]

For \( \Delta^0_{D,jk} \neq 0 \), this can be expanded as

\[
\frac{\Delta^0_{D,jk,lm}}{\Delta_D^0} = \frac{\Delta^0_{D,jk,lm}}{\Delta^0_{D,jk}} \left[ \begin{array}{c} \Delta^0_{D,jk} \\ \Delta^0_{D,lm} \end{array} \right].
\]

Now \( \Delta^0_{D,jk} / \Delta^0_D \) is just \( R^0_{jk} \), as shown previously, but the first factor’s meaning has not yet been established. Note that for \( g_{jk} \neq 0 \)

\[
\Delta^0_{D,jk} = \lim_{g_{jk} \to \infty} \frac{\Delta^0_{D,jk,lm}}{g_{jk}}.
\]

If \( g_{jk} \) is allowed to approach infinity, the last term vanishes, the first term on the right-hand side remains finite and nonzero, and the left-hand side is unaffected. Therefore,

\[
\Delta^0_{D,jk} = \lim_{g_{jk} \to \infty} \frac{\Delta^0_{D,jk,lm}}{g_{jk}}.
\]

A similar analysis applies if row \( l \) and column \( m \) are deleted, so

\[
\Delta^0_{D,jk,lm} = \lim_{g_{jk} \to \infty} \left[ \begin{array}{c} \Delta^0_{D,jk} \\ \Delta^0_{D,lm} \end{array} \right].
\]

Thus

\[
\Delta^0_{D,jk,lm} = \lim_{g_{jk} \to \infty} \left[ \begin{array}{c} \Delta^0_{D,jk} \\ \Delta^0_{D,lm} \end{array} \right] = R^0_{ml} = \lim_{g_{jk} \to \infty} \frac{g_{jk}}{g_{jk}}.
\]

It is possible to contrive circuits for which \( \Delta^0_{D,jk,lm} \) and \( a_{jk, lm} \neq 0 \) in such cases, the relation \( a_{jk, lm} = \lim_{g_{jk} \to \infty} R^0_{ml} / g_{jk} \) can be used to find \( a_{jk, lm} \).

In general, calculation of \( R^0_{ml} \) requires that the transresistance from port \( j \) to port \( m \) be calculated for a test circuit in which a voltage-controlled current-source (VCCS) of value \( g_{jk} V_c \) is placed at port \( j \). The resulting expression is then evaluated in the limit \( g_{jk} \to \infty \). Fortunately, with many models, such a VCCS already exists in parallel with each transcapacitor, so the expression for \( R^0_{ml} \) already includes the needed terms for \( g_{jk} \), and no additional test circuit need be evaluated. For the case of a reciprocal capacitor \( C_{ij}, R^0_{ij} \) is found by computing \( R^0_{ml} \) for a circuit with port \( j \) shorted.

Note that the roles of index pairs \( lm \) and \( jk \) are completely symmetrical, so the coefficients of \( C_{jk} R^0_{lm} \) can be computed in either of two ways:

\[
a_{jk, lm} = R^0_{lm} R^0_{jk} = R^0_{ml} R^0_{jk}.
\]

The second-order coefficient can be computed using

\[
a_s = \sum_{C_D} C_{jk, lm} a_{jk, lm}.
\]

The third-order term is based on the set \( C_{D3} \) of all unique products of triplets \( C_{jk} C_{lm} C_{ij} \) of nonzero capacitive elements in \( C_D \) such that \( j \neq i \), \( j \neq p \), and \( j \neq k \), and \( m \neq p \), and \( m \neq q \), and \( q \neq k \). The restrictions eliminate triplets with two or more elements in the same row (a capacitor and a transcapacitor or two transcapacitors at the same port), and those with elements in the same column (a capacitor and a TCV or two TCVs at the same port). Each triplet has a corresponding coefficient, which is easily shown to be

\[
a_{jk, lm, pq} = R^0_{jk} R^0_{lm} R^0_{pq} = a_{jk, lm} R^0_{pq}
\]

where \( R^0_{pq} = \lim_{g_{pq} \to \infty} [R^0_{pq}] \). The products of all the triplets in \( C_{D3} \) and their corresponding coefficients can be summed to form \( a_s \).

This procedure can then be generalized and used to calculate denominator terms through \( a_{N} \). The order of the denominator is usually less than \( N_x \), since many of the higher order terms are zero.

**Determination of the Numerator Polynomial**

The numerator of the system function can be written as

\[
\Delta_N = \Delta_{0,0} = h_0 + h_1 z + h_2 z^2 + \cdots h_N z^N
\]

where \( N \) is the order of the numerator.

The constant term \( h_0 \) is the dc limiting value of the transfer function

\[
h_0 = \Delta^0_{0,0} = H(0) = \lim_{t \to 0} H(s).
\]
TABLE II
SUMMARY OF TRANSFER FUNCTION COEFFICIENT CALCULATIONS
FOR THE EXTENDED TIME-CONSTANT METHOD

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{jk} = R_{j1} )</td>
<td>( j = k = m )</td>
</tr>
<tr>
<td>( a_{jk,lm} = R_{j1} )</td>
<td>( j = l, j &lt; q, l &lt; q, k = m, k &lt; p, m = p )</td>
</tr>
</tbody>
</table>

By analogy to (9), \( \Delta_{jk,lm} \) can be written as

\[
\Delta_{jk,lm} = \lim_{g_{jk} \to \infty} \frac{\Delta_{0,lm}}{\Delta_{0,jk}}. \tag{17}
\]

If \( a_{jk} = \Delta_{D,jk} / \Delta_{D} \neq 0 \), from (17), \( \Delta_{D} = \Delta_{D,jk}g_{jk} \), which means that \( \Delta_{D} \) does not depend on \( g_{jk} \). Thus \( \Delta_{D} \) is finite. This is equally true if \( C_{jk} \) was eliminated from set \( C_{D} \).

The second-order coefficient is based on the set \( C_{N2} \) of all unique products of pairs \( C_{jk}C_{lm} \) of nonzero capacitive elements in \( C_{N} \) such that \( j \neq l \) and \( k \neq m \). If \( \Delta_{D,jk},lm \) is nonzero and both \( C_{jk} \) and \( C_{lm} \) are elements of \( C_{D} \),

\[
h_{jk,lm} = \lim_{g_{jk} \to \infty} \frac{[H^{0}]R_{ml}R_{kj}}{[H^{0}]R_{mk}R_{kj}} = H^{0}\Delta_{D,jk,lm} g_{jk}. \tag{20}
\]

If \( C_{jk} \) or \( C_{lm} \) is not an element of \( C_{D} \), or if \( a_{jk,lm} = 0 \), then

\[
\lim_{g_{jk} \to \infty} \Delta_{D,jk,lm} = 0
\]

so that

\[
h_{jk,lm} = \lim_{g_{jk} \to \infty} \frac{[H^{0}]}{g_{jk}}. \tag{21}
\]

Generalization to higher order coefficients is straightforward.

The results of this section are summarized in Table II.

III. EXAMPLE: COMMON-SOURCE AMPLIFIER

An example will help to clarify the extended time-constant technique. The circuit shown in Fig. 1 represents a common-source amplifier with source degeneration. The transistor model includes two transcapacitors, \( C_{41} \) and \( C_{42} \), in parallel with \( g_{m} \) and \( g_{mb} \) VCCS's. The circuit with all capacitive elements removed is analyzed to find the voltage transfer function. The results are given in Table III.

All of the capacitive elements contribute to the first-order time constant. Coefficient \( a_{41} = R_{41}^{0} \) is found by applying a current source \( I_{4} \) at port 4 in the same direction as the \( C_{41} \) controlled source; \( R_{41}^{0} \) is found as \( V_{1} / I_{4} \). \( a_{42} \) is found by a similar technique.

![Common-source amplifier equivalent circuit.](image)
There are 10 unique combinations of the five capacitive elements taken in pairs; of these, pairs $C_{14}C_{41}$, $C_{22}C_{42}$, and $C_{41}C_{42}$ are excluded from the set $C_{D}$, since they each share common ports. The remaining second-order coefficients are easy to evaluate. All, both of which are just $C_{3}$, are excluded since it is in parallel with the output port. Most of the remaining coefficients all evaluate to zero. Similarly, $R_{33}$ can be found as $\lim_{\omega \to 0} R_{33}$ and $R_{23}$ is $\lim_{\omega \to 0} R_{23}$, both of which are just $R_D$. Therefore,

$$a_{33,41} = a_{41} R_{41}^{[1]} = a_{33,42},$$

$$a_{42} R_{33}^{[1]} = R_D R_S / [1 + (g_m + g_{mb}) R_S].$$

(22)

Similar analysis shows that $a_{11,33}$ and $a_{22,33}$ give the same result. Although there are 10 ways to form triplets of the capacitive elements, all but three are eliminated from set $C_{D}$ because of shared ports. The remaining coefficients all evaluate to zero. The leading coefficient $h_0$ of the numerator polynomial is the dc value of the transfer function:

$$h_0 = H^0 = -g_m R_D / [1 + (g_m + g_{mb}) R_S].$$

(23)

The set $C_{D}$ needed to find the first-order coefficient $h_1$ includes only the four elements $C_{11}$, $C_{22}$, $C_{31}$, and $C_{41}$. $C_{22}$ is excluded since it is in parallel with the output port. Most of the terms needed to find $h_1$ can be calculated by inspection. For example, $H^4$ is just $\lim H^0$ so that $h_0 = 0$. There are three pairs of elements in set $C_{D}$: $C_{15}C_{22}$, $C_{14}C_{42}$, and $C_{22}C_{42}$. Since $h_4 = 0$, $h_{1,4} = 0$. In the other two cases, since the corresponding denominator coefficient is finite, (21) must be used. Since $H^4$ is finite, $h_{22,41}$ can be found from

$$r_{22,41} = \lim_{\omega \to 0} \left[ \frac{H^0}{R_{22} R_{41}} \right] = \lim_{\omega \to 0} \left[ \frac{R_D H^{41}}{g_m} \right].$$

(24)

The complete transfer function can be written as

$$V_2(x) = \left[ g_{m} + (g_{m} C_{22} + g_{m} C_{41} - g_{m} C_{11}) R_S \right] R_D,$$

$$V_1(x) = A + B x + C x^2$$

where

$$A = 1 + (g_m + g_{mb}) R_S,$$

$$B = R_S \left[ C_{11} + C_{22} + C_{41} + C_{42} - C_{31} \{ (g_m + g_{mb}) R_S \} \right],$$

$$C = R_D \left[ C_{11} C_{22} + C_{14} C_{42} + C_{22} C_{42} + C_{31} C_{42} \right].$$

IV. CONCLUSION

This work has demonstrated an extension of the time-constant method to allow for nonreciprocal capacitive elements. A large number of cross products often arise in the higher order terms of transfer functions. Adding transcapacitors to the transistor model complicates the expressions even more. For this reason, the most common use of the extended time-constant method is likely to be to provide an easy way to gain an intuitive feel for the effects of transcapacitive elements on circuit behavior.

ACKNOWLEDGMENT

The authors would like to thank John Prentice of Harris Corporation for pointing out early work in this field.